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LETTER TO THE EDITOR

Factorisation of operators and coupled nonlinear evolution equations

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Abstract. Pairs of coupled nonlinear evolution equations are considered. Factorisation of the L-operator that relates the equations in such pairs may be realised by the operators of creation and annihilation of solitons, and yields the unitary scattering operator for the processes of collision of arbitrary wave disturbances with solitons as well as a new form of the Bäcklund transformation.

We consider the equations (Korteweg and de Vries 1895, Toda 1967)

$$u_t + 6uu_x - u_{xxx} = 0 \tag{1}$$

$$(\ln v_k)_t = i_{k-1/2} - i_{k+1/2}, \qquad (i_k)_t = v_{k-1/2} - v_{k+1/2} \tag{2}$$

which belong to the Lax class of exactly solvable equations (Lax 1968)

$$L_t = LA - AL. \tag{3}$$

The remarkable feature of this class of equations is the existence of soliton solutions which behave like one-dimensional particles. Each soliton corresponds to one discrete eigenvalue of the L-operator. This fact allows us to use the quantum mechanical factorisation method (Infeld and Hill 1951, Green 1965), which is a conventional tool for investigating the characteristics of the discrete spectrum of operators.

The present Letter extends the factorisation method to investigate the soliton dynamics for nonlinear evolution equations.

Consider a factorisation of the L-operator for equations (1) and (2),

$$L = H^*H - \alpha^2, \tag{4}$$

where * means conjugation, and H is given by

$$H = \tilde{u} + \mathrm{d}/\mathrm{d}x \tag{5}$$

$$H = \exp(\tfrac{1}{2}\,\mathrm{d}/\mathrm{d}k)\tilde{v}_k + \tilde{i}_k \exp(-\tfrac{1}{2}\,\mathrm{d}/\mathrm{d}k) \tag{6}$$

for (1) and (2) respectively. Representation (4) allows us to build the following equations coupled with (1) and (2):

$$\tilde{u}_t + 6\tilde{u}^2\tilde{u}_x - \tilde{u}_{xxx} = 0 \tag{7}$$

$$(\ln \tilde{v}_k)_t = \tilde{i}_{k-1/2} - \tilde{i}_{k+1/2}, \qquad (\ln \tilde{i}_k)_t = \tilde{v}_{k-1/2} - \tilde{v}_{k+1/2}. \tag{8}$$

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For these equations, expression (3) may be realised by the (L, A)-pair

$$\tilde{L}=\begin{pmatrix} 0 & H^* \\ H & 0 \end{pmatrix}, \qquad \tilde{A}=\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \tag{9}$$

where A_1 and A_2 have the form of the A-operator for the initial equations and depend on

$$u_1=\tilde{u}^2+\tilde{u}_x-\alpha^2, \qquad u_2=\tilde{u}-\tilde{u}_x-\alpha^2 \tag{10}$$

and

$$\begin{aligned} i_{1k}&=\tilde{i}_k+\tilde{v}_{k+1/2}, & v_{1k}&=\tilde{i}_k\tilde{v}_{k-1/2} \\ i_{2k}&=\tilde{v}_k+\tilde{i}_{k+1/2}, & v_{2k}&=\tilde{v}_k\tilde{i}_{k-1/2} \end{aligned} \tag{11}$$

for equations (7) and (8) respectively. Expressions (10) and (11) follow directly from the representation

$$\tilde{L}^2_t=\tilde{L}^2\tilde{A}-\tilde{A}\tilde{L}^2$$

and are recognised as the generalised Miura transformations (GMTs) which connect two solutions of one nonlinear equation (1) or (2) with solutions of a new equation (7) or (8). These transformations can be written in the operator form

$$L_1=H^*H-\alpha^2, \qquad L_2=HH^*-\alpha^2. \tag{12}$$

Elimination in these equalities of the solution with the 'tilde' yields the expression which relates directly two different solutions of (1) or (2), or the so-called 'spatial part' of the Bäcklund transformation (BT).

An essential moment of the further consideration is the fact that GMTs have the form of the Riccati equation for solutions with a 'tilde'. This allows us to consider three different solutions distinguished by asymptotes at $x\to\pm\infty$. For instance, for (10)

$$\tilde{u}_{\text{in}}(\pm\infty, t)=-\alpha, \qquad \tilde{u}_{\text{out}}(\pm\infty, t)=\alpha, \qquad \tilde{u}(\pm\infty, t)=\pm\alpha.$$

As will be shown below, the indices 'in' and 'out' which are introduced here relate these three solutions to the problem of collision with solitons.

Such ambiguity of GMTs yields three possible factorisations of the L-operator:

$$L_1=H^*_{\text{in}}H_{\text{in}}-\alpha^2=H^*_{\text{out}}H_{\text{out}}-\alpha^2=H^*H-\alpha^2, \tag{13}$$

$$L_{2\text{in}}=H_{\text{in}}H^*_{\text{in}}-\alpha^2, \tag{14}$$

$$L_{2\text{out}}=H_{\text{out}}H^*_{\text{out}}-\alpha^2, \tag{15}$$

$$L_2=HH^*-\alpha^2. \tag{16}$$

The potential u_2 of the operator $L(u_2)$ has the following asymptotic form at $t\to\mp\infty$ (Perel'man *et al* 1974),

$$u_2\to\begin{cases} u_{\text{in}}-2\alpha^2\,\text{sech}^2\,\alpha x, & t\to-\infty \quad (17) \\ -2\alpha^2\,\text{sech}^2\,(\alpha x-\delta)+u_{\text{out}}, & t\to\infty, \quad (18) \end{cases}$$

and describes the process of collision of disturbance u_{in} with a soliton for equation (1). The potentials $u_{2\text{in}}$ and $u_{2\text{out}}$ of the operators $L(u_{2\text{in}})$ and $L(u_{2\text{out}})$ have asymptotic forms

$$u_{2\text{in}}\to u_{\text{in}}, \qquad t\to-\infty \tag{19}$$

$$u_{2\text{out}}\to u_{\text{out}}, \qquad t\to\infty. \tag{20}$$

In order to solve the collision problem (17), (18), it is necessary to express u_{out} in terms of u_{in}.

We determine the unitary scattering operator S,

$$\Psi_{2\text{out}} = S\Psi_{2\text{in}}, \tag{21}$$

where

$$L_{2\text{in}}\Psi_{2\text{in}} = k^2\Psi_{2\text{in}}, \qquad L_{2\text{out}}\Psi_{2\text{out}} = k^2\Psi_{2\text{out}},$$

$$\tilde{L}\tilde{\Psi} = \lambda\tilde{\Psi}, \qquad \tilde{\Psi} = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}, \qquad k^2 = \lambda^2 - \alpha^2.$$

From (14), (15) and (21), the next equivalent definition follows,

$$H_{\text{out}} = SH_{\text{in}},$$

or, in the explicit form,

$$S = H_{\text{out}}WH^*_{\text{in}} \tag{22}$$

where W is the weight operator, which in the representation of eigenfunctions of the operator $L(u)$ is λ^{-2}.

According to Fridman and El'yaserich (1976), $\tilde{u}$ may be expressed in terms of u_2 as a series of α^{-1} powers. Consider $\tilde{u}$ at $t \to \mp\infty$:

$$\tilde{u} \to \begin{cases} \tilde{u}_{\text{in}} + \alpha \tanh \alpha X, & t \to -\infty \\ \alpha \tanh(\alpha x - \delta) + \tilde{u}_{\text{out}}, & t \to \infty. \end{cases}$$

Thus for equation (7) we have the process of scattering of disturbance $\tilde{u}_{\text{in}}$ with a shock wave, the scattering operator for which is related to the operator (22),

$$\tilde{S}_{\text{sh}} = \begin{pmatrix}1 & 0\\ 0 & S\end{pmatrix}; \tag{23}$$

i.e.

$$\tilde{\Psi}_{\text{out}} = \tilde{S}_{\text{sh}}\tilde{\Psi}_{\text{in}}, \tag{24}$$

where

$$\tilde{L}_{\text{in}}\tilde{\Psi}_{\text{in}} = \lambda\tilde{\Psi}_{\text{in}}, \qquad \tilde{L}_{\text{in}} = \begin{pmatrix}0 & H^*_{\text{in}}\\ H_{\text{in}} & 0\end{pmatrix},$$

$$\tilde{L}_{\text{out}}\tilde{\Psi}_{\text{out}} = \lambda\Psi_{\text{out}}, \qquad \tilde{L}_{\text{out}} = \begin{pmatrix}0 & H^*_{\text{out}}\\ H_{\text{out}} & 0\end{pmatrix}.$$

In conclusion, the scattering operator for collision $\tilde{u}_{\text{in}}$ with a soliton of equation (7) is

$$\tilde{S}_{\text{sol}} = \begin{pmatrix}S_1 & 0\\ 0 & S_2\end{pmatrix}, \tag{25}$$

where S_1 and S_2 describe scattering disturbances $u_{1\text{in}}$ and $u_{2\text{in}}$, which are defined by (10), with a soliton for equation (1).

The investigation which has been undertaken for equations (1) and (7) is fully applicable to (2) and (8). The scattering operators S, S_{sh} and S_{sol} are built from H and H^* defined by formula (6).

An action of the operators $H_{\rm in}$, $H_{\rm out}$ and H on the eigenfunctions of operators $L(\tilde{u}_1)$ and $L(u_2)$ may be analysed best of all in terms of scattering data (or through the variables 'angle–action'), since it is defined by an asymptotic behaviour of these operators. Consider, for example, the scattering data for operators $L(u_1)$,

$$\{S(k);\ k_i,\ \mu_i^2(t)\} \tag{26}$$

$$\{S(z);\ z_i,\ \mu_i^2(t)\}, \tag{27}$$

for equations (1) and (2) respectively, where $S(k)$ is the reflection coefficient, k_i, z_i the discrete eigenvalues, and $\mu_i^2(t)$ the normalization of eigenfunctions. Then the action of $H_{\rm in}$ and $H_{\rm out}$ on (26) and (27) yields the scattering data for operators $L(u_{2\rm in})$ and $L(u_{2\rm out})$,

$$\left\{S(k)\left(\frac{\alpha+\mathrm{i}k}{\alpha-\mathrm{i}k}\right)^{1/2};\ k_i,\ \mu_i^2(t)\frac{\alpha-k_i}{\alpha+k_i}\right\} \tag{28}$$

$$\left\{S(k)\left(\frac{\alpha-\mathrm{i}k}{\alpha+\mathrm{i}k}\right)^{1/2};\ k_i,\ \mu_i^2(t)\frac{\alpha+k_i}{\alpha-k_i}\right\} \tag{29}$$

$$\begin{aligned}\Bigg\{S(z)\frac{\cosh(\phi+\mathrm{i}\phi_z/2)}{\cosh(\phi-\mathrm{i}\phi_z/2)};\ z_i,\ \mu_i^2(t)\frac{\cosh(\phi+\phi_i/2)}{\sinh(\phi+\phi_i/2)\sinh(\phi-\phi_i/2)},\qquad & i=1,\ldots,N_1;\\ z_j,\ \mu_j^2(t)\frac{\cosh^2(\phi+\phi_j/2)}{\cosh^2(\phi-\phi_j/2)},\qquad j=N_1+1,.u.,N\Bigg\}\end{aligned} \tag{30}$$

$$\begin{aligned}\Bigg\{S(z)\frac{\cosh(\phi-\mathrm{i}\phi_z/2)}{\cosh(\phi+\mathrm{i}\phi_z/2)};\ z_i,\ \mu_1^2(t)\frac{\cosh^2(\phi-\phi_i/2)}{\sinh(\phi+\phi_i/2)\sinh(\phi-\phi_i/2)},\qquad & i=1,\ldots,N_1;\\ z_j,\ \mu_j^2(t)\frac{\mathrm{cossh}^2(\phi-\phi_j/2)}{\cosh^2(\phi+\phi_j/2)},\qquad j=N_1+1,\ldots,N\Bigg\},\end{aligned} \tag{31}$$

where

$$(v_n)_{\rm in}\to \mathrm{e}^{\phi},\qquad |n|\to\infty;\qquad (v_n)_{\rm out}\to\mathrm{e}^{-\phi},\qquad |n|\to\infty;$$

$$\arg z=\mathrm{e}^{\phi_z},\qquad \operatorname{mod} z_j=\mathrm{e}^{\phi_j},$$

$$\arg z_j=\begin{cases}2k\pi, & j=1,\ldots,N_1\\ (2k+1)\pi, & j=N_1+1,\ldots,N.\end{cases}$$

It is clear that the scattering operator S transforms the scattering data (28) and (30) into (29) and (31) respectively.

The situation is different when the operator H acts on the scattering data (26) and (27). This operator adds one discrete eigenvalue to the spectrum. Introducing the index n, which describes the number of solitons in the system, we have

$$H_{n+1}\Psi_n=\lambda\Psi_{n+1}$$

$$H^*_{n+1}\Psi_{n+1}=\lambda\Psi_n,$$

where the operators H_n and H^*_n may be interpreted as the operators of creation and annihilation of solitons. This allows us to construct the spatial part of the BT for equations (7) and (8):

$$H_nH^*_n-\alpha_n^2=H^*_{n+1}H_{n+1}-\alpha^2_{n+1}. \tag{32}$$

It is interesting to note that unlike the GMT or BT for equations (1) or (2), formula (32) couples the H-operators with different indices n.

Besides, it follows from (28), (30) and (29), (31) that for scattering with solitons the variables 'action' do not change while the variables 'angle' acquire a jump.

The treatment given above allows the consideration of the following equations:

$$v_t + v^2 v_x + v_{xxx} = 0 \tag{33}$$

$$(N_k)_t = N_k(N_{k+1} - N_{k-1}). \tag{34}$$

In these cases it is more convenient to factorise the square of the L-operators. In particular, for equation (33) L^2 may be expressed as

$$L = H^*H - \alpha^2, \qquad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & \tilde{v} \\ \tilde{v}^* & 0 \end{pmatrix}. \tag{35}$$

For equation (34) L^2 represents the L-operator for the Toda (1967) lattice, and factorisation leads to the equation

$$(\tilde{N}_k)_t = (\lambda^2 - \tilde{N}_k^2)(\tilde{N}_{k+1} - \tilde{N}_{k-1}). \tag{36}$$

Further consideration of equations (33) and (34) is almost exactly the same as discussed above.

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